Resit exam — Partial Differential Equations (WBMA008-05)

Thursday 11 July 2024, 15.00h-17.00h

University of Groningen

Instructions

1. The use of calculators is *not* allowed. It is allowed to use a "cheat sheet" (one sheet A4, both sides, handwritten,"wet ink").
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (8 + 6 + 6 = 20 points)

Consider the following nonuniform transport equation:

$$\frac{\partial u}{\partial t} + \frac{\sqrt{1+x^2}}{x+\sqrt{1+x^2}}\frac{\partial u}{\partial x} = 0, \quad u(0,x) = \cos(3\pi x).$$

(a) Show that the characteristic curves are given by the equation

$$x + \sqrt{1+x^2} = t + k, \quad k \in \mathbb{R}.$$

(b) Compute the value of the solution u at the point $(t,x) = (-\sqrt{2}, 0)$.

(c) Is the solution u at the point $(t,x) = (1+\sqrt{2}, 1)$ determined by the initial condition?

Problem 2 (15 + 5 = 20 points)

Consider the following wave equation with Dirichlet boundary conditions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(t,0) = u(t,1) = 0.$$

(a) Determine all nontrivial solutions of the form $u(t,x) = w(t)v(x)$.

(b) Compute the solution that satisfies $u(0,x) = 3\sin(x) - \sin(2x)$ and $u_t(0,x) = 0$.

Problem 3 (4 + 6 = 10 points)

(a) Show that $u(x,y) = \cos(x)\cosh(y)$ is a harmonic function.

(b) Compute the integral $\displaystyle\int_{-\pi}^{\pi} \cos(\pi + 2024\cos(t))\cosh(2024\sin(t))\,dt$.

Turn page for problems 4 and 5!

Problem 4 (20 points)

Recall the following function:

$$G_0(x,y;\xi,\eta) = -\frac{1}{2\pi}\log\|(x,y)-(\xi,\eta)\|,$$

where $\|\cdot\|$ denotes the Euclidean norm. Use the method of images to construct Green's function for Poisson's equation on the following domain:

$$\Omega = \{(x,y) \in \mathbb{R}^2 : x > 0,\ -x < y < x\}.$$

Hint: for a given point $(\xi,\eta) \in \Omega$ consider the image points (η,ξ), $(-\eta,-\xi)$, and $(-\xi,-\eta)$.

Problem 5 (20 points)

Consider the following equation:

$$\frac{\partial u}{\partial t} = e^{-t}\frac{\partial^2 u}{\partial x^2}, \quad u(0,x) = f(x), \quad t > 0, \quad -\infty < x < \infty.$$

Use Fourier transforms to find a solution of the form

$$u(t,x) = \int_{-\infty}^{\infty} G(t,x-\xi)f(\xi)\,d\xi$$

and determine an explicit expression for the function G (i.e. without using integrals).

End of test (90 points)

Solution of problem 1 (8 + 6 + 6 = 20 points)

(a) Define the function

$$\beta(x) := \int \frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\,dx = \int 1+\frac{x}{\sqrt{1+x^2}}\,dx = x+\sqrt{1+x^2}.$$

(4 points)

The characteristic curves are given by $\beta(x) = t+k$, where $k \in \mathbb{R}$ is a constant, or, equivalently,

$$x+\sqrt{1+x^2} = t+k.$$

(4 points)

(b) The point $(-\sqrt{2},0)$ lies on the characteristic curve for $k = 1+\sqrt{2}$.
(2 points)

This curve intersects the x-axis in the point $(0,1)$.
(2 points)

Since the solution is constant along characteristic curves we have

$$u(-\sqrt{2},0) = u(0,1) = \cos(3\pi) = -1.$$

(2 points)

(c) The point $(1+\sqrt{2},1)$ lies on the characteristic curve for $k = 0$.
(2 points)

But the curve $x+\sqrt{1+x^2} = t$ never intersects the x-axis. Indeed, the left hand side is positive for all $x \in \mathbb{R}$. Therefore, the value of the solution at the point $(1+\sqrt{2},1)$ is not determined by the initial condition.
(4 points)

Solution of Problem 2 (15 + 5 = 20 points)

(a) Substituting $u(t,x) = w(t)v(x)$ in the equation gives

$$\frac{w''(t)}{w(t)} = \frac{v''(x)}{v(t)} = \lambda$$

where λ is a constant, and the boundary conditions on u imply that $v(0) = v(1) = 0$.
(1 point)

Case 1: $\lambda = 0$. In this case we have $v(x) = a + bx$ and the boundary conditions imply that $a = b = 0$ which only results in trivial solutions.
(3 points)

Case 2: $\lambda = \omega^2 > 0$. In this case we have $v(x) = a\cosh(\omega x) + b\sinh(\omega x)$ and the boundary conditions imply that $a = b = 0$ which only results in trivial solutions.
(3 points)

Case 3: $\lambda = -\omega^2 < 0$. In this case we have $v(x) = a\cos(\omega x) + b\sin(\omega x)$ and the boundary conditions imply that $a = 0$ and $b\sinh(\omega) = 0$. For nontrivial solutions we need that $\omega = k\pi$ where $k \in \mathbb{N}$.
(6 points)

The solution of the resulting w-equation is $w(t) = a\cos(k\pi t) + b\sin(k\pi t)$, where a, b are arbitrary constants. Therefore, all possible nontrivial solutions are of the form

$$u_k(t,x) = \big[a_k\cos(k\pi t) + b_k\sin(k\pi t)\big]\sin(k\pi x), \quad k \in \mathbb{N}.$$

(2 points)

(b) The superposition principle gives the general solution

$$u(t,x) = \sum_{k=1}^{\infty}\big[a_k\cos(k\pi t) + b_k\sin(k\pi t)\big]\sin(k\pi x).$$

(1 point)

By comparing coefficients it follows that the solution that satisfies the initial conditions $u(0,x) = 3\sin(\pi x) - \sin(2\pi x)$ and $u_t(0,x) = 0$ is given by

$$u(t,x) = 3\cos(\pi t)\sin(\pi x) - \cos(2\pi t)\sin(2\pi x).$$

(4 points)

Solution of problem 3 (4 + 6 = 10 points)

(a) We have the following partial derivatives:

$$\begin{aligned} u_x &= -\sin(x)\cosh(y), \\ u_{xx} &= -\cos(x)\cosh(y), \\ u_y &= \quad \cos(x)\sinh(y), \\ u_{yy} &= \quad \cos(x)\cosh(y). \end{aligned}$$

We obtain $u_{xx} + u_{yy} = 0$, which shows that u is harmonic.
(4 points)

(b) Let C be the circle with center $(\pi, 0)$ and radius $r = 2024$. By the mean value property for harmonic functions we have

$$\frac{1}{2\pi} \oint_C u\,ds = u(\pi, 0).$$

(3 points)

Therefore, we have that

$$\int_{-\pi}^{\pi} \cos(\pi + 2024\cos(t))\cosh(2024\sin(t))\,dt = 2\pi u(\pi, 0) = -2\pi.$$

(3 points)

Solution of problem 4 (20 points)

We construct the Green's function by setting $G = G_0 + z$, where the function z satisfies $\Delta z = 0$ on Ω and $z = -G_0$ on $\partial\Omega$ (or, equivalently, $G = 0$ on $\partial\Omega$). To a point $(\xi,\eta) \in \Omega$ we associate three image points $(\xi_k, \eta_k) \in \mathbb{R}^2 \setminus \overline{\Omega}$ where $k = 1,2,3$. The ansatz

$$z(x,y;\xi,\eta) = \sum_{i=1}^{3} \frac{a_k}{2\pi} \log \|(x,y) - (\xi',\eta')\| + \frac{b_k}{2\pi}.$$

guarantees that z is harmonic on Ω.
(4 points)

The points on $\partial\Omega$ are given by (x,x) and $(x,-x)$ for $x \geq 0$. For all $x \geq 0$ we have the following distances between the boundary points and the image points:

$$\|(x,x) - (\xi,\eta)\| = \sqrt{(x-\xi)^2 + (x-\eta)^2} = \|(x,x) - (\eta,\xi)\|,$$

$$\|(x,x) - (-\eta,-\xi)\| = \sqrt{(x+\xi)^2 + (x+\eta)^2} = \|(x,x) - (-\xi,-\eta)\|,$$

$$\|(x,-x) - (\xi,\eta)\| = \sqrt{(x-\xi)^2 + (x+\eta)^2} = \|(x,-x) - (-\eta,-\xi)\|,$$

$$\|(x,-x) - (\eta,\xi)\| = \sqrt{(x+\xi)^2 + (x-\eta)^2} = \|(x,-x) - (-\xi,-\eta)\|.$$

(8 points)

From this we conclude that we can set $b_k = 0$ and $a_k = \pm 1$ for $k = 1,2,3$.
(2 points)

This gives the following candidate:

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\big[-\log\|(x,y) - (\xi,\eta)\| \\ \pm \log\|(x,y) - (\eta,\xi)\| \pm \log\|(x,y) - (-\eta,-\xi)\| \pm \log\|(x,y) - (-\xi,-\eta)\|\big].$$

We can determine the correct plus and minus signs as follows. By substituting (x,x) or $(x,-x)$ for (x,y) and requiring that $G = 0$ on $\partial\Omega$ it follows that

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\big[-\log\|(x,y) - (\xi,\eta)\| \\ + \log\|(x,y) - (\eta,\xi)\| + \log\|(x,y) - (-\eta,-\xi)\| - \log\|(x,y) - (-\xi,-\eta)\|\big].$$

(6 points)

Solution of problem 5 (20 points)

Taking the Fourier transform of both sides of the equation gives

$$\frac{d\widehat{u}}{dt} = -e^t k^2 \widehat{u}.$$

The solution is given by

$$\widehat{u}(t,k) = \widehat{u}(0,k)e^{-e^t k^2} = \widehat{f}(k)e^{-e^t k^2}.$$

(4 points)

Write $\widehat{g_t}(k) = e^{-e^t k^2}$. Then

$$u(t,x) = \frac{1}{\sqrt{2\pi}}(g_t * f)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g_t(x-\xi)f(\xi)\,d\xi.$$

(4 points)

From the table of Fourier transforms we obtain

$$\mathscr{F}\left[e^{-ax^2}\right] = \frac{e^{-k^2/(4a)}}{\sqrt{2a}}.$$

In particular, setting $a = e^{-t}/4$ gives

$$\mathscr{F}\left[e^{-e^{-t}x^2/4}\right] = \sqrt{2}e^{t/2}e^{-e^t k^2}.$$

(4 points)

Taking the inverse transform gives

$$g_t(x) = \mathscr{F}^{-1}\left[e^{-e^t k^2}\right] = \frac{1}{\sqrt{2}}e^{-t/2}e^{-e^{-t}x^2/4}.$$

(4 points)

In conclusion, by setting

$$G(t,x) = \frac{1}{\sqrt{2\pi}}g_t(x) = \frac{1}{2\sqrt{\pi}}e^{-t/2}e^{-e^{-t}x^2/4},$$

we obtain the desired formula

$$u(t,x) = \int_{-\infty}^{\infty} G(t,x-\xi)f(\xi)\,d\xi.$$

(4 points)